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On the correspondence between fermionic number and statistics of solitons

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Abstract: Solitons of a nonlinear field interacting with fermions often acquire a fermionic number or an electric charge if fermions carry a charge. We establish a correspondence between charge and statistics (or spin) of solitons showing how the same mechanism (chiral anomaly) gives solitons statistical and rotational properties of fermions. These properties are encoded in a geometrical phase, i.e., an imaginary part of a euclidian action for a nonlinear σ-model. In the most interesting cases the geometrical phase is non-perturbative and has a form of an integer-valued theta-term.

Keywords: Sigma Models, Nonperturbative Effects, Solitons Monopoles and Instantons, Anomalies in Field and String Theories.

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Contents

1. Introduction

The idea that a soliton of a nonlinear field interacting with fermions may acquire quantum numbers proportional to its own topological charge has a long history. The first step is traditionally attributed to T.H.R. Skyrme who forty years ago [1] suggested that baryons emerge as solitons of a meson field. In 1968 Finkelstein and Rubinstein [2] showed that a soliton may be a fermion. The phenomenon became widely recognized probably after 1976 seminal papers of Jackiw and Rebbi [3], P. Hasenfratz and G. ‘t Hooft [4], and A. S. Goldhaber [5] who showed that a monopole interacting with Dirac fermions acquires spin 1/2 through a zero mode of Dirac equation. In condensed matter physics G. Fröhlich already in 1954 knew that in 1D electronic system kinks of phonon modes carry an electric charge [6]. Later the appearance of soliton quantum numbers has been linked to anomalous properties of current algebras [7]. The papers which shaped the subject can be found in the reprint volume [8]. Nowadays the phenomena when spatial topological configurations of a nonlinear bose field acquire fermionic charges and through them rotational and statistical properties of fermions are ubiquitous.

Fermionic number, spin, and statistics of solitons emerge differently. The first can be probed for by an external gauge field. The second and the third emerge through a geometrical phase of the wave function or, equivalently, through a topological term in an effective bosonic action [9, 10]. The geometrical phase changes by 2π times a quantum number (spin or statistics) in an adiabatic motion along a closed path in the configurational space which corresponds to a full rotation or to an interchange of solitons respectively. A relation between the topological charge of a soliton and its spin/statistics is well established (see [11, 12] and references therein).

Intuitively one expects that once a soliton acquires fermionic charge it must inherit fermionic rotational properties as well. In other words there must be a correspondence between soliton's fermionic charge and its spin (statistics). This is indeed true although a relation between these two aspects remains relatively obscure. In the most interesting cases, when the system is explicitly time-reversal invariant in odd space-time dimensions or chiral invariant in even space-time dimensions, the geometrical phase is a theta-term, i.e., an integer-valued topological invariant. In this case it does not appear in perturbation theory and had been often missed in analysis. Contrary, the fermionic charge can always be found using perturbative methods [13]. We address this issue in this letter. We argue that once a soliton acquired a fermionic charge the corresponding geometrical phase must be present in the nonlinear sigma model. [1] We use some results of our recent paper [14]. For an earlier discussion see [12, 15] and references therein.

To simplify our analysis we concentrate on a particular example of a sigma model in $(2+1)$ dimensions. Extension to more general cases is relatively straightforward but more mathematically involved. We do not attempt it in this publication.

2. Fermionic σ-models

Fermionic σ-models [2] describe Dirac fermions interacting with a nonlinear field $\hat{n}$:

$$\mathcal{L} = \bar{\psi}(i\hat{D} + im \cdot \hat{n})\psi \,. \tag{2.1}$$

Here $\hat{D} = \gamma_\mu(\partial_\mu - iA_\mu)$, where A_μ is an external Abelian gauge field (e.g., electromagnetic field), γ_μ are Dirac gamma-matrices, and $\hat{n}$ is a matrix field which belongs to some compact target space V. We use a euclidian formulation with $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. Fermions are also form an isomultiplet on which matrix $\hat{n}$ acts. Here and below we write the action in the euclidian spacetime.

The nature of a geometrical phase depends on the topology of a target space and of a spacetime. In order to illustrate how a soliton with a unit topological charge and a unit fermionic charge becomes a relativistic particle with a spin $1/2$ we choose a particular target space $V = CP^M$ and a three-dimensional $(2+1)$

[1] Let us stress a difference between the well understood relation of spin and topological charge of a soliton and a relation between the fermionic number of a soliton and its spin.

[2] Physical applications of fermionic σ-models are known. In one spatial dimension, e.g., it is a familiar Peierls-Fröhlich model of polyacetylene [6]. The model in $(3+1)$ appeared in the past as a model for nuclear forces [7]. All these models also emerge in the context of topological superfluids [16].

spacetime [15, 17]. It is important that for any integer M static configurations of $\hat{n}$ fall into homotopy classes $\pi_2(CP^M) = Z$. This means that among configurations of $\hat{n}$ there are topologically non-trivial ones — solitons.

To make sure that matrix $\hat{n}$ belongs to CP^M target space [18] we parameterize explicitly $\hat{n} = 2zz^\dagger - 1$, where $z = (z_1, z_2, \ldots, z_{M+1})^t$ is a complex $M+1$-component unit vector $z^\dagger z = 1$. One can easily check that $(M+1) \times (M+1)$ matrix $\hat{n}$ satisfies constraint $\hat{n}^2 = 1$ and does not change when z is multiplied by a phase factor. This means that the target space is CP^M. In the special case $M = 1$, the target space becomes two-dimensional sphere $CP^1 = S^2$ and $\hat{n} = \vec{n}\vec{\tau}$, where $\vec{n} = (n_1, n_2, n_3) = z^\dagger \vec{\tau} z$ is a unit vector $\vec{n}^2 = 1$ and $\vec{\tau}$ is a set of three Pauli matrices acting in isospace.

There is an important difference between CP^1 model and CP^M model with $M > 1$. The former has a symmetry with respect to a time reversal (or parity) transformation $x_0 \to -x_0$, supplemented by a a π rotation in the target space $\vec{n} \to -\vec{n}$. Contrary, the CP^M model does not have this symmetry: the change $\hat{n} \to -\hat{n}$ can not be achieved by $SU(M+1)$-transformation within the target space. The time reversal or parity is explicitly broken and is not a symmetry of the problem for $M > 1$.

3. Nonlinear σ-models

The nonlinear σ-model is a result of an integration over fermions

$$e^{-\mathcal{W}[n]} = \int D\psi D\bar{\psi}\, e^{-\int dx \mathcal{L}} = \mathrm{Det}(i\hat{D} + im \cdot \hat{n})\,. \tag{3.1}$$

For smooth and slow configurations of n-field the effective action is local and can be systematically studied in $1/m$ (gradient) expansion. In a euclidian spacetime it has an imaginary part $\mathcal{W} = \mathrm{Re}\,\mathcal{W} + i\,\mathrm{Im}\,\mathcal{W}$. The leading terms of the gradient expansion of a nonlinear σ-model (we compute them later) are

$$\mathrm{Re}\,\mathcal{W} = \frac{m}{16\pi} \int d^3x \;\mathrm{tr}\;(\partial_\mu \hat{n})^2, \tag{3.2}$$

$$\mathrm{Im}\,\mathcal{W} = -\pi\,\Gamma - \int d^3x\; A_\mu J_\mu, \tag{3.3}$$

where the *topological current* J_μ is defined as

$$J_\mu = -\frac{i}{16\pi} \epsilon_{\mu\nu\lambda}\,\mathrm{tr}\;(\hat{n}\partial_\nu \hat{n} \partial_\lambda \hat{n}) \tag{3.4}$$

and $\pi\,\Gamma$ stays for the *geometrical phase.*

The two terms of an imaginary part (3.3) of the effective action reflect different quantum numbers of solitons. They are of the topological origin, do not depend on any scale, and remain in the action even in the limit of infinite mass $m \to$

∞. The second term in (3.3) is a response to an external field. It establishes a relation [13, 15, 19] between the fermionic current in the background n-field $j_\mu = i\langle\bar{\psi}\gamma_\mu\psi\rangle$ and the topological current J_μ

$$j_\mu = i\frac{\delta}{\delta A_\mu}\mathcal{W} = J_\mu + O(1/m)\,. \tag{3.5}$$

It represents a well-known fact that a soliton acquires fermionic charge due to a chiral anomaly. The total number of fermionic levels with positive energy changes by Q if the topological charge of the vacuum changes by Q

$$\int_{\text{space}} d^2r\ j_0 = Q \equiv \int_{\text{space}} d^2r\ J_0\,. \tag{3.6}$$

Note that although eq. (3.5) has higher gradient corrections, eq. (3.6) is exact: a soliton with a topological charge Q carries fermionic number Q.

Let us assume that the coordinate space is compactified into a two-dimensional sphere S^2. Then a smooth static configuration of bosonic field $\hat{n}(r)$ defines a mapping of S^2 into the target space $V = CP^M$. All mappings $S^2 \to CP^M$ can be divided into distinct topological (homotopy) classes so that any two configurations within one class can be smoothly deformed one into another. Classes are enumerated by an integer number Q — the topological charge, which in this case can be any integer number. The integer Q can be expressed as a spatial integral of the density of topological charge $Q = \int d^2r\ J_0(r)$ where J_0 is a zeroth component of the conserved topological current (3.4).[3]

Because the topological current is local, one can think about localized topological solitons. In particular a special configuration with zero net topological charge which consists of two well-separated positive and negative bumps of unit topological charge can be considered as a separated soliton-antisoliton pair.

Let us now turn to the first term in (3.3) — the geometrical phase. It represents the change of the phase of a fermionic wave function due to a change of the background configuration of $\hat{n}$. The following spacetime configurations are of special interest: (i) the adiabatic 2π-rotation of a single localized soliton around some axis and (ii) an adiabatic interchange of two solitons.[4] If a soliton has spin S and θ-statistics, the phase of the fermionic wave function (the geometrical phase) changes in these processes by $2\pi S$ and θ respectively, where $\theta = \pi, 0$ for a fermion (boson).

[3] The mechanism discussed in the paper holds for a fermionic model (2.1) in d+1 dimensions with both non-trivial d-homotopy group $\pi_d(V) = Z$ and d-cohomology group of the target space $H_d(V) = Z$. The latter means that there exists a topological current — closed but not exact d-form.

[4] In a spacetime compactified into a sphere, the total number of solitons is zero. In this case, in order to probe for spin of solitons one needs an adiabatic process in which soliton-antisoliton pair is created, then separated in space, the soliton is rotated by 2π-angle, and then the pair is annihilated [9].

We will show that if soliton acquired a unit of a fermionic charge, described by the second term in (3.3), it should also acquire rotational numbers of a fermion, i.e., spin $S = 1/2$ and fermionic statistics $\theta = \pi$, described by the first term in (3.3). This corresponds to the change of the phase $\pi\,\Gamma$ by π for both processes.

4. Calculation of fermionic number

We start from the calculation of the fermionic number of a soliton — the second term in (3.3). This computation is perturbative, well known [13, 15, 19], and constitutes the chiral anomaly. We sketch it below.

A variation of an effective action with respect to a gauge field A_μ is $\delta_A \mathcal{W} = -\mathrm{tr}\,\delta_A D D^{-1} = -\mathrm{tr}\left\{\delta\not{A}\mathcal{D}^\dagger(\mathcal{D}\mathcal{D}^\dagger)^{-1}\right\}$, where $\mathcal{D} = i\hat{D} + im\hat{n}$, $\mathcal{D}^\dagger = i\hat{D} - im\hat{n}$, and $\mathcal{D}\mathcal{D}^\dagger = (i\not\partial + \not{A})^2 + m^2 + m\not\partial\hat{n}$. Expanding the denominator in gradients of chiral fields and keeping only terms linear in A in the imaginary part of the action,[5] we obtain

$$\delta_A \,\mathrm{Im}\,\mathcal{W} = \mathrm{tr}\left\{\delta\not{A}\hat{n}\frac{m}{-\partial^2 + m^2}\left(\not\partial\hat{n}\frac{m}{-\partial^2 + m^2}\right)^2\right\}. \tag{4.1}$$

Calculating the functional trace and the trace over 2×2 gamma-matrices, we obtain eq. (3.5): the fermionic current is equal to the topological current (3.4).

5. Geometrical phase and topology of target space

Spacetime configurations of chiral bosonic field $\hat{n}(r,t)$ define mappings of a spacetime into the target space V. Here we consider the spacetime compactified into a sphere S^3 so that we have mappings $S^3 \to V$.[6] We have to distinguish between two possibilities. The first is when $\pi_3(V) = 0$, i.e., any two of spacetime configurations (mappings) can be smoothly deformed one into another. In this case the geometrical phase is perturbative. The second possibility is when $\pi_3(V) = Z \neq 0$ and configurations are divided into topological (homotopy) classes. The CP^M model in $(2+1)$ realizes both cases: $\pi_3(CP^M) = 0$ at $M > 1$ and $\pi_3(CP^1) = \pi_3(S^2) = Z$. In the latter case noncontractable spacetime configurations are labeled by the Hopf number. We will see that the geometrical phase in this case is equal to the Hopf number (multiplied by π). In this case the geometrical phase in non-perturbative.

[5] We consider here only terms linear in the external gauge field A and omit the Chern-Simons term present in CP^M models (with $M > 1$) because of an explicitly broken parity/time reversal symmetry. This Chern-Simons term is important but does not play a role in this analysis.

[6] A more physical compactification of spacetime is the one into the product space $S^2 \times S^1$. In this case an explicit expression for a geometrical phase is much more involved. In particular the compactification into product spaces leads to some restrictions on the allowed spin and statistics of solitons. However, it does not change our conclusions (see e.g., a discussion and references in [14]).

To calculate the geometrical phase we put A_μ to zero and calculate a variation of an effective action over the chiral field $\hat{n}$

$$\begin{aligned}
\delta \operatorname{Im} \mathcal{W} &= -\operatorname{Im} m \operatorname{tr}\left(m \delta \hat{n} \mathcal{D}^\dagger (\mathcal{D}\mathcal{D}^\dagger)^{-1}\right) \\
&= -i \operatorname{tr}\left\{ m \delta \hat{n} \hat{n} \frac{m}{-\partial^2 + m^2} \left(\not{\partial} \hat{n} \frac{m}{-\partial^2 + m^2} \right)^3 \right\} \\
&= \frac{1}{32\pi} \int_{S^3} d^3 x \, \epsilon_{\mu\nu\lambda} \operatorname{tr}\left(\delta \hat{n} \hat{n} \partial_\mu \hat{n} \partial_\nu \hat{n} \partial_\lambda \hat{n}\right) .
\end{aligned} \tag{5.1}$$

First, consider the case when all spacetime configurations of $\hat{n}$ are contractable: $\pi_3(V) = 0$. In this case one can restore the geometrical part of the action from its variation (5.1). For $V = CP^M$ we write $\hat{n} = 2zz^\dagger - 1$ and (after some algebra) we obtain for $M \neq 1$

$$\delta \operatorname{Im} \mathcal{W} = \frac{1}{4\pi} \int_{S^3} d^3 x \, \delta(\epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda) \,, \tag{5.2}$$

where a “gauge field” is $a_\mu \equiv -i z^\dagger \partial_\mu z$. This gives the geometrical phase

$$\Gamma = -\frac{1}{4\pi^2} \int_{S^3} d^3 x \, \epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \,. \tag{5.3}$$

The geometrical phase is perturbative. It changes under a small variation of the field $\hat{n}$. Under the time reversal transformation the geometrical phase (5.3) changes sign. This reflects the fact that the model (2.1), at $M > 1$ is not invariant under the time reversal.

This method does not work, however, in the most interesting case when $\pi_3(V) \neq 0$ and the time reversal symmetry holds. In this case the variation of the geometrical phase over chiral field is identically zero. E.g., for CP^1 model $\epsilon_{\mu\nu\lambda} \operatorname{tr}(\delta \hat{n} \hat{n} \partial_\mu \hat{n} \partial_\nu \hat{n} \partial_\lambda \hat{n}) = 0$. However, this does not mean that the geometrical phase vanishes. It vanishes perturbatively, but in fact it is proportional to a topological invariant (the Hopf number) labeling the class of a mapping from $\pi_3(V)$.

The strategy to go around the topological obstacle has been suggested by Witten [11, 20]. It is known as a method of embedding [21]. The idea is to extend the fermionic σ-model with the target space V to a one with a bigger space $\tilde{V}$, so that any mapping of a spacetime into $\tilde{V}$ is topologically trivial $\pi_3(\tilde{V}) = 0$. Then the geometrical phase varies under the variation of $\hat{n}$, such that (5.1) does not vanish and one can use the perturbative result (5.1) to restore the geometrical phase. Then the proper reduction to the target space of interest $\tilde{V} \to V$ should be made in such a fashion that (2.1) becomes the model of interest plus decoupled free fermions. Under this reduction the geometrical phase becomes an integer-valued topological invariant. This method was applied to CP^1 model[17] with an extension to a topologically trivial $\tilde{V} = CP^M$ model. The result for the geometrical phase is given by the same eq. (5.3) as for $M > 1$. For $M = 1$, however, it is an integer-valued Hopf

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invariant [22]. The geometrical phase in this case is invariant (mod(2π)) under the same time reversal transformation as the original fermionic model (2.1) at $M = 1$. This is a general situation: the geometrical phase of the system invariant under the time reversal in odd space-time dimensions is an integer valued topological invariant (a theta-term).

To calculate the geometrical phase we compare the effective action for two physically different configurations of chiral field, e.g., corresponding to a soliton at rest and to a rotating soliton. These configurations belong to different topological classes and, therefore, it is impossible to interpolate between them smoothly. Singular interpolating configurations require an ultraviolet regularization of the theory. The embedding method essentially provides us with such a regularization. It allows to connect topologically different configurations by a smooth interpolation through an extended target space.[7]

Some consequences of the geometrical phase in the model with $M = 1$ have been discussed in [16]. For a search of stable field configurations with nonzero Hopf numbers see [23].

6. Charge-spin-statistics correspondence

Now we are ready to show that under a rotation of a localized soliton, or when interchanging positions of two solitons, the geometrical phase changes by π in accord with the unit fermionic charge of a soliton. This establishes a *charge-spin-statistics correspondence* in addition to the usual relation between spin/statistics and the topological charge of a soliton: if a soliton possesses fermionic number Q, it also has the spin Q/2. Respectively it has the fermionic (bosonic) statistics for Q odd (even). The topological charge of the soliton depends on the number of relevant fermionic species. For one species it is Q.

Let us consider a soliton which is slowly rotated by an angle 2π and compare the geometrical phase of this process with the one without rotation. For simplicity we choose a static localized soliton $\hat{n}(r)$ with a spherically symmetric topological charge density $J_0(r)$. Topological current written in terms of the gauge field a_μ is

[7] The embedding method we use is similar to the computation of global anomaly in gauge theories with Weyl fermions [20, 21]. However, there is an essential difference. In gauge theories with Weyl fermions the value of an effective action is compared for two physically identical field configurations which differ by a large (noncontractable) gauge transformation. To compare these effective actions, the two gauges were interpolated by a non-pure-gauge (physical) configuration of the gauge field. The result was that the effective actions calculated for "identical" configurations differ by a phase. The global anomaly signals an inconsistency of theory with Weyl fermions unless the content of the theory is fine tuned so that the geometrical phase difference is a multiple of 2π. In the context of the nonlinear sigma model, we discuss in this paper, the geometrical phase differs for physically different configurations. In this case the geometrical phase is necessary for the consistency of the theory and leads to observable effects (see, e.g., [11, 20, 16]).

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$J_\mu = \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda$. We choose a vector potential $a_\mu = (a_0, \vec{a})$, such that $a_0 = 0$, and $\vec{\nabla} \times \vec{a} = 2\pi J_0(r)$. Then the vector potential $a_\mu = (\omega(t)(q(r) - 1), \vec{a})$ where $q(r) = \int_0^r dr' 2\pi r' J_0(r')$ corresponds to a soliton rotating with the angular velocity $\omega(t)$. The topological current is $J_\mu = (J_0(r), J_0(r)\omega\epsilon^{ab}r_b)$. The geometrical phase (5.3) of the rotating soliton is $-\frac{1}{4\pi^2}\int d^2r\, 2a_0\vec{\nabla} \times \vec{a} = \frac{\omega}{2\pi}Q$ which gives $\Gamma = Q\int dt\, \frac{\omega(t)}{2\pi} = Q\frac{\Delta\phi}{2\pi}$. A 2π-rotation of soliton gives rise to the phase $\pi\Gamma = \pi Q$. This corresponds to the spin $Q/2$ of the soliton.[8] Similar arguments can be applied for the process of an interchange of positions of two solitons.

In the perturbative case the change of the geometrical phase, say under rotation, can be calculated directly from its variation (5.1). An adiabatic rotation of a soliton can be treated as a small variation of the background field. The result, however, is the same for both perturbative and nonperturbative cases.

7. Conclusion

To summarize, we argued that as soon as a soliton of a nonlinear field acquires a fermionic number as a result of an interaction with fermions, the soliton acquires also all other fermionic quantum numbers: statistics and spin. These quantum numbers are represented by a geometrical phase in the effective action of the nonlinear σ-model. In some interesting cases when the configuration space has noncontractable paths, the geometrical phase becomes an integer-valued θ-term. We discussed the both cases on the example of $2+1$-dimensional theory, where a single species of fermions interact with a chiral bose field taking values in CP^M target space. However, the fermionic number-spin-statistics correspondence is a general phenomenon. An extension of our results to more general cases: arbitrary number of fermionic species, other target spaces, and dimensions (both odd or even) is relatively straightforward [14].

In general, we formulate (in physical terms) the following "theorem". If (i) mappings of the coordinate space into a target space of a nonlinear bose field are noncontractable[9] and (ii) a noncontractable static configuration carries a fermionic number as a result of an interaction with fermions, then the effective action of the bose field possesses a geometrical phase which changes by π times fermionic number of a soliton under 2π-spatial rotation of a localized soliton. If solitons are localized the same phase appears in a process of interchanging of positions of solitons.

[8] Similar arguments used in [12, 11] to establish the connection between the topological charge of a Skyrmion and its spin.

[9] In fact we need not only $\pi_d(G) \neq 0$. Instead, one should require that $H_d(V) \neq 0$ and to avoid the multitude of lower-dimensional topological terms one should also require that $\pi_i(G) = 0$ for all $i < d$. Then the existence of noncontractable spatial configurations of bose field ($\pi_d(G) \neq 0$) is a consequence of Gurevich theorem (see e.g., [22]).

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References

[1] T.H.R. Skyrme, *A unified field theory of mesons and baryons*, *Nucl. Phys.* **31** (1962) 556.

[2] D. Finkelstein and J. Rubinstein, *Connection between spin, statistics and kinks*, *J. Math. Phys.* **9** (1968) 1762.

[3] R. Jackiw and C. Rebbi, *Spin from isospin in a gauge theory*, *Phys. Rev. Lett.* **36** (1976) 1116.

[4] P. Hasenfratz and G.'t Hooft, *A fermion-boson puzzle in a gauge theory*, *Phys. Rev. Lett.* **36** (1976) 1119.

[5] A.S. Goldhaber, *Spin and statistics connection for charge-monopole composites*, *Phys. Rev. Lett.* **36** (1976) 1122.

[6] G. Fröhlich, *On the theory of superconductivity: the one-dimensional case.*, *Proc. R. Soc.* **A223** (1954) 296.

[7] S. B. Treiman, R. Jackiw, B. Zumino, and E. Witten, *Current algebra and anomalies*, Princeton University Press, Princeton 1985 and references therein.

[8] C. Rebbi and G. Soliani, *Solitons and particles*, World Scientific 1984.

[9] F. Wilczek and A. Zee, *Linking numbers, spin and statistics of solitons*, *Phys. Rev. Lett.* **51** (1983) 2250.

[10] A.M. Polyakov, *Fermi-Bose transmutations induced by gauge fields*, *Mod. Phys. Lett.* **A 3** (1988) 325.

[11] E. Witten, *Current algebra, baryons and quark confinement*, *Nucl. Phys.* **B 223** (1983) 433.

[12] A. P. Balachandran, *Skyrmions* in TASI High Energy Physics 1985, M.J. Bowick and F. Gürsey eds., World Scientific, Singapore.

[13] J. Goldstone and F. Wilczek, *Fractional quantum numbers on solitons*, *Phys. Rev. Lett.* **47** (1981) 986.

[14] A.G. Abanov and P.B. Wiegmann, *Theta-terms in nonlinear sigma models*, *Nucl. Phys.* **B 570** (2000) 685 [hep-th/9911025].

[15] T. Jaroszewicz, *Induced fermion current in the σ model in (2+1)- dimensions*, *Phys. Lett.* **B 146** (1984) 337; *Induced topological terms, spin and statistics in* $(2+1)$ *dimensions*, *Phys. Lett.* **B 159** (1985) 299; *Fermion-induced spin of solitons: vacuum and collective aspects*, *Phys. Lett.* **B 193** (1987) 479

[16] A.G. Abanov and P.B. Wiegmann, *Chiral non-linear σ models as models for topological superconductivity*, *Phys. Rev. Lett.* **86** (2001) 1319 [hep-th/0006157].

[17] A.G. Abanov, *Hopf term induced by fermions*, *Phys. Lett.* **B 492** (2000) 321 [hep-th/0005150].
See also the second reference in [15]

[18] For construction of CP^M model see, e.g., A. D'Adda, M. Luscher and P. Di Vecchia, *A $1/N$ expandable series of nonlinear sigma models with instantons*, *Nucl. Phys.* **B 146** (1978) 63.

[19] For review see: A.J. Niemi and G.W. Semenoff, *Fermion number fractionization in quantum field theory*, *Phys. Rep.* **135** (1986) 99.

[20] E. Witten, *Global aspects of current algebra*, *Nucl. Phys.* **B 223** (1983) 422.

[21] S. Elitzur and V.P. Nair, *Nonperturbative anomalies in higher dimensions*, *Nucl. Phys.* **B 243** (1984) 205;
see also F.R. Klinkhamer, *Another look at the* SU(2) *anomaly*, *Phys. Lett.* **B 256** (1991) 41.

[22] M. Monastyrsky and O. Efimov, *Topology of gauge fields and condensed matter*, Plenum Pub., 1993.

[23] L.D. Faddeev and A.J. Niemi, *Toroidal configurations as stable solitons*, hep-th/9705176.